(Hyper) Graph Partitioning

Past & Present.

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Joint work w/...

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Based on : arxiv / 2301.08920

About this talk...

PART 1 : A past result.

→ Provide a new proof for a known fast graph partitioning algorithm using convex optimization tools.

PART 2 : A present application

→ Show how this view yields new approx. algorithms for hypergraph partitioning.

Part 3: A future direction (time permitting)

→ Future applications of these tools?

Part I : The Past

Let's begin...

FINDING CUTS OF MINIMUM EXPANSION

__INPUT:__ $G = (V, E, w^G)$, $w_e^G \geq 0 \ \forall e \in E$.

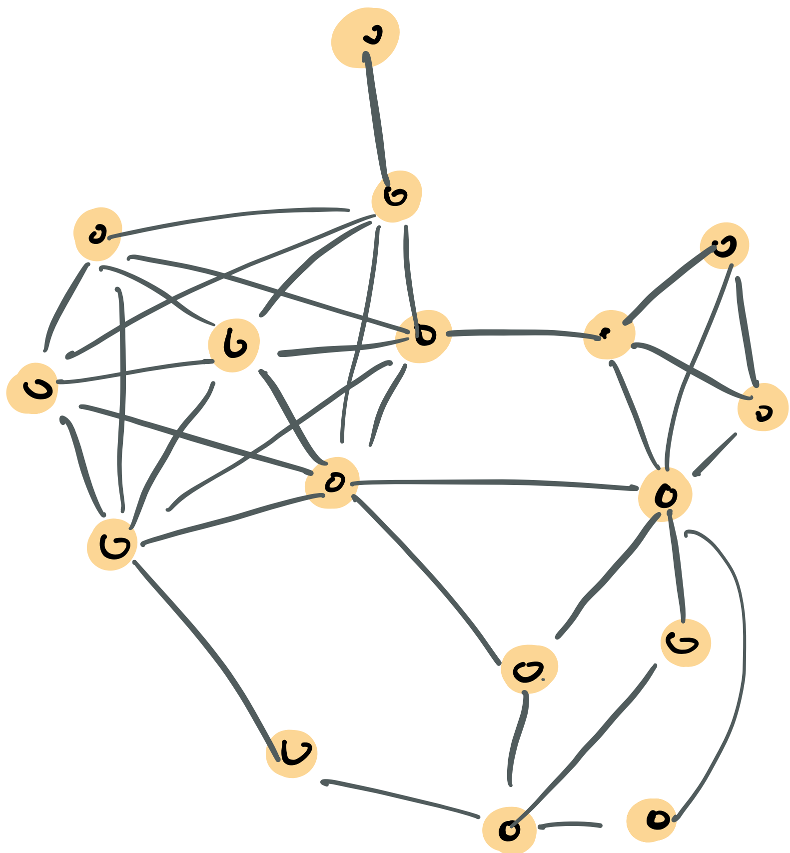
__OUTPUT:__ $S \subseteq V$ cut minimizes for,

$$\phi_G(S) = \frac{e_G(S, V \setminus S)}{\min\{|S|, |V \setminus S|\}}$$

$$\longrightarrow \quad \phi_G = \min_{S \subseteq V} \phi_G(S)$$

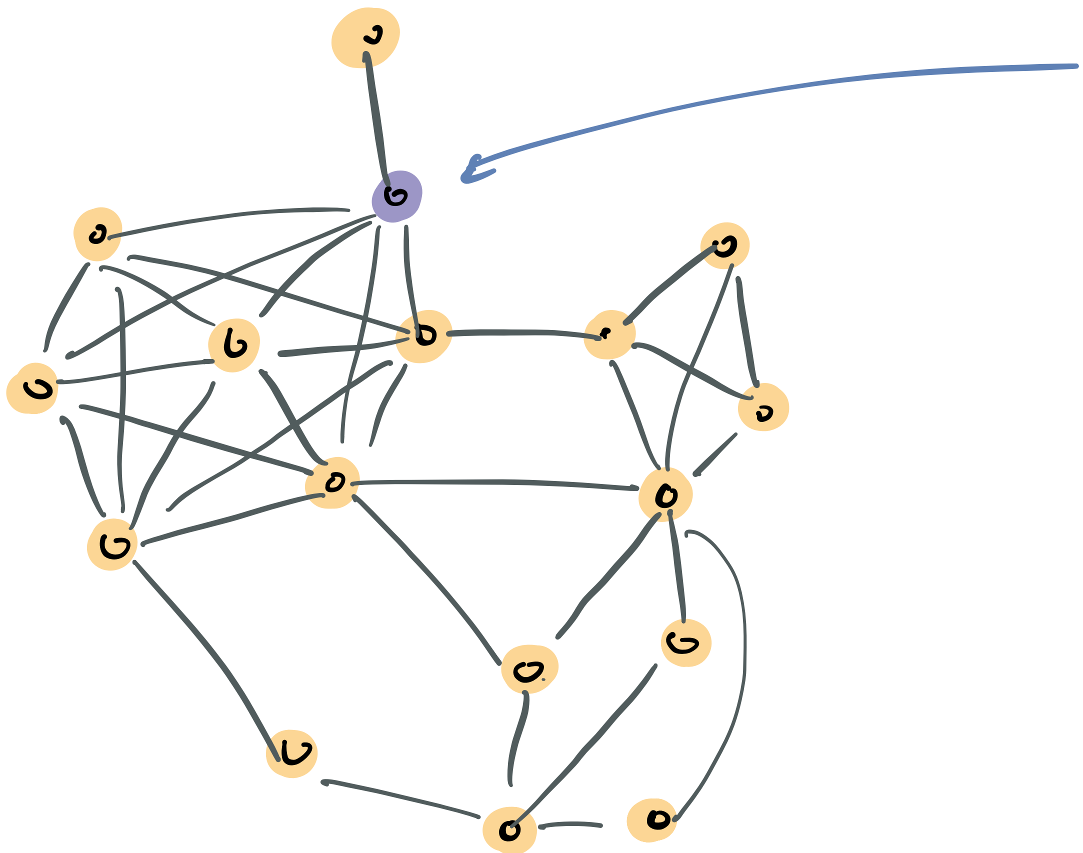
$e_G(S, V \setminus S) \Rightarrow$ wtd sum across edges crossing $(S, V \setminus S)$.

For example...

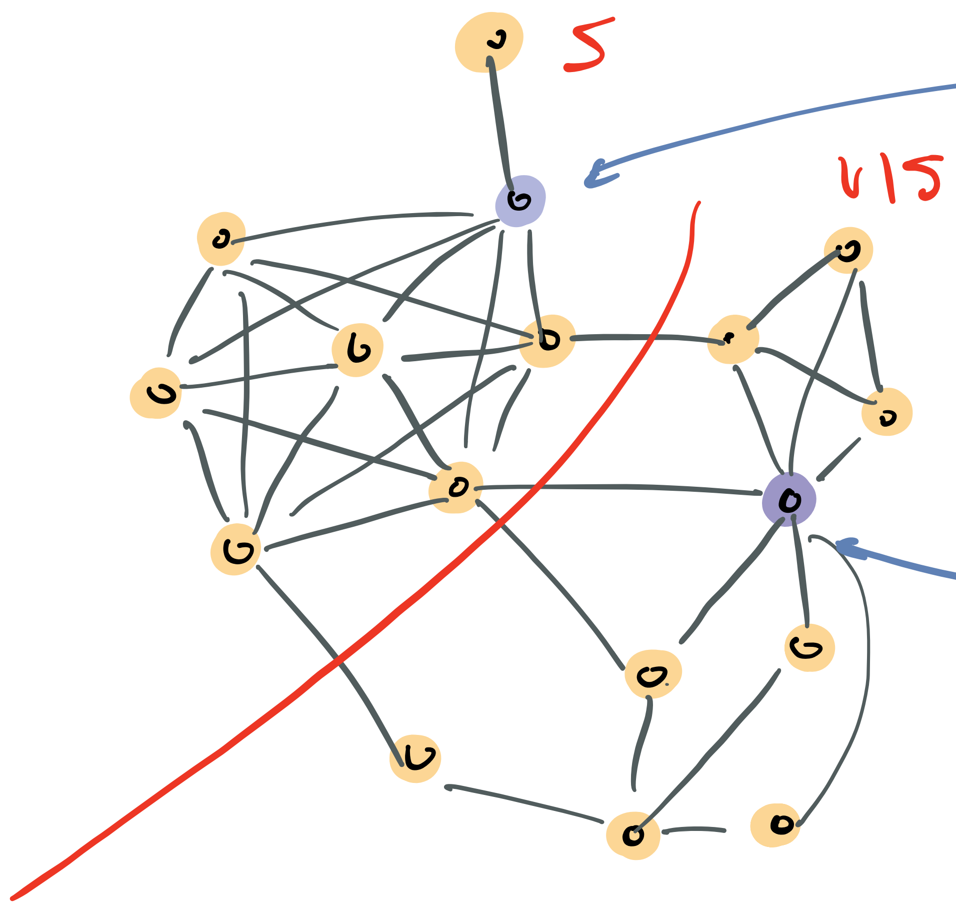


Friend network

For example . . .


Elizabeth

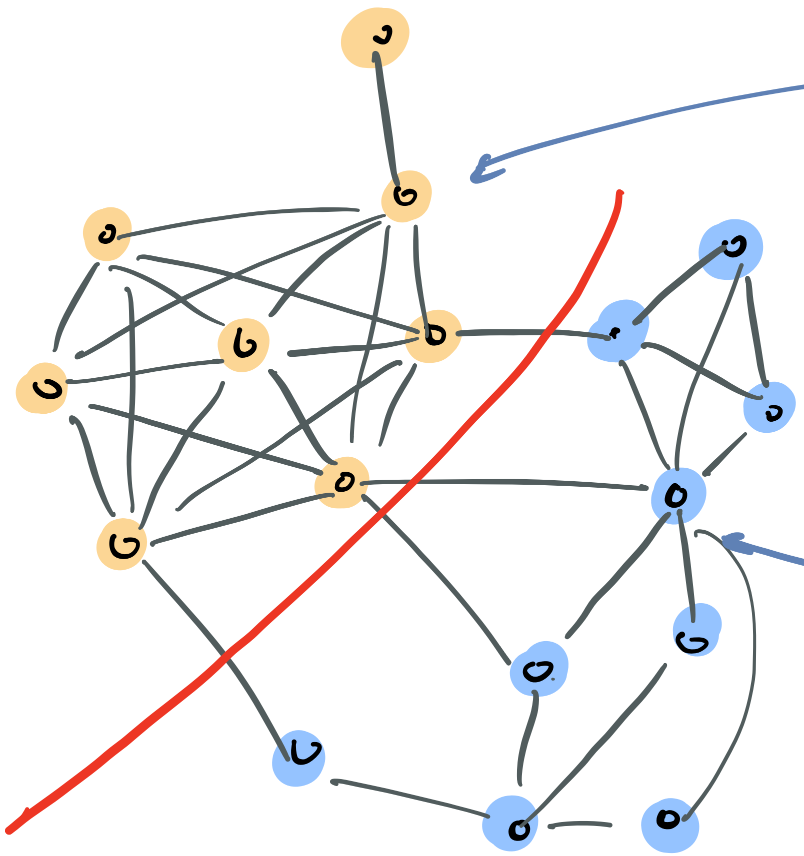
For example...



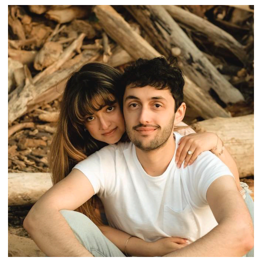
Elizabeth

Jonathan

For example...

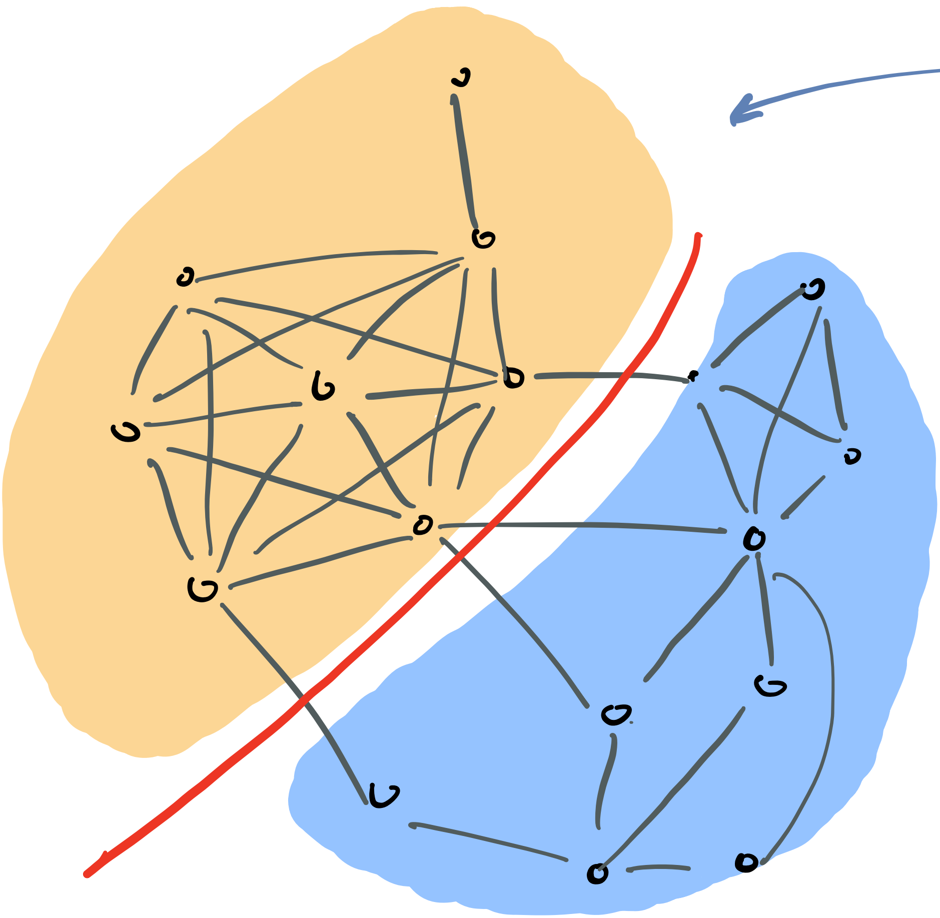


Elizabeth



Jonathan

For example...



Friends from MD

Friends from Berkeley.

Finding minimum expansion cuts is ...

- NP-hard → Want polynomial time approximation algorithms... But!

WANT TO WORK OVER MASSIVE DATASETS.

algorithm runs in time subquadratic w.r.t. size of graph

Overall Goal: construct an algorithm that outputs an $O(\text{polylog}(n))$-approximate minimum expansion cut using $O(\text{polylog}(n))$ maximum flows

Theorem [CKLPGS22]: Exact max flow in almost-linear $O(m^{1+o(1)})$-time ⚡

$\longrightarrow$ Almost-linear approx for expansion.

And... we already know how to do this...

<u>Theorem</u> [KRV06]: There exists an algorithm which outputs an $O(\log^2 n)$-approximation using $O(\log^2 n)$ maximum flow computations.

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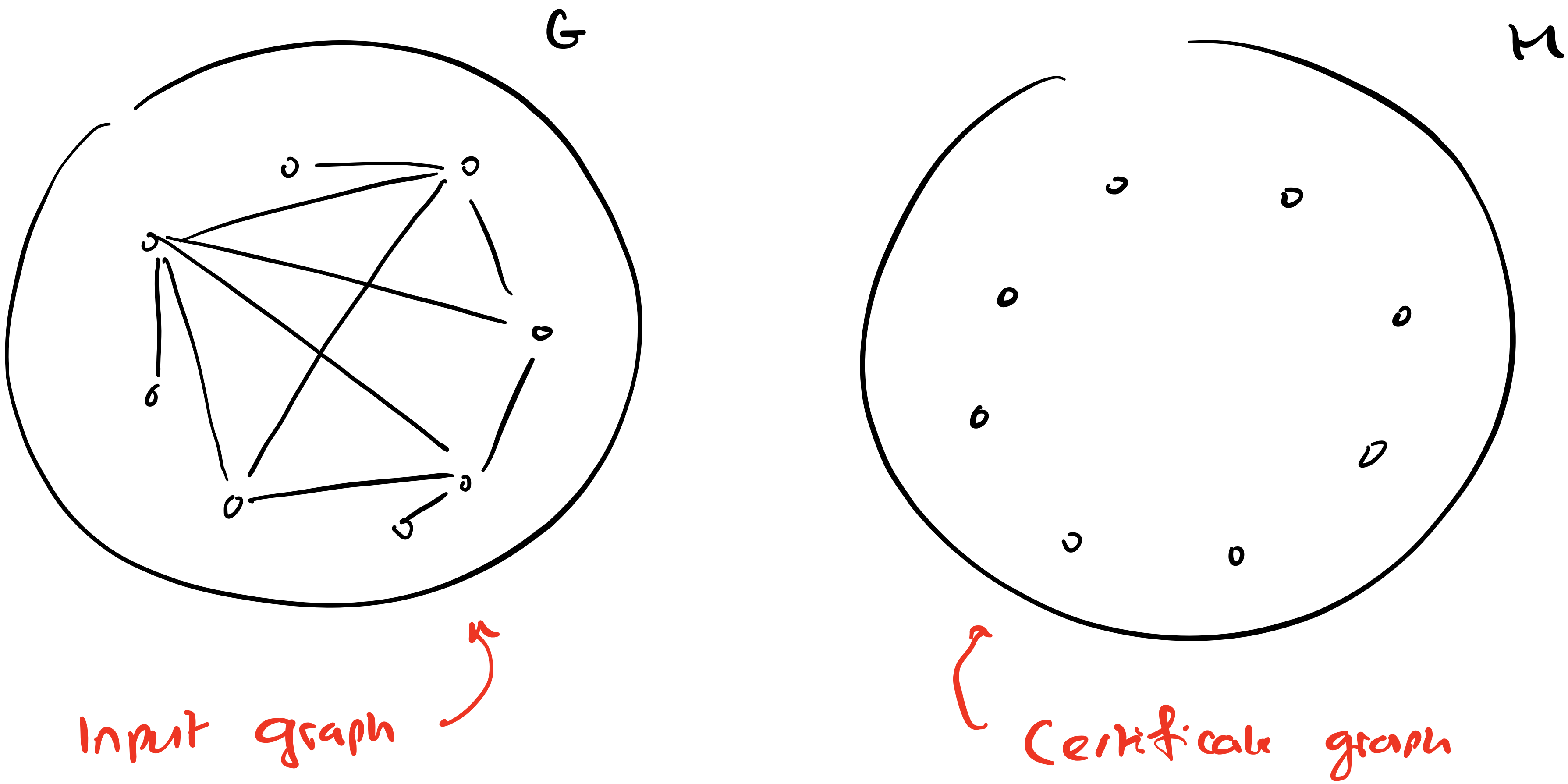
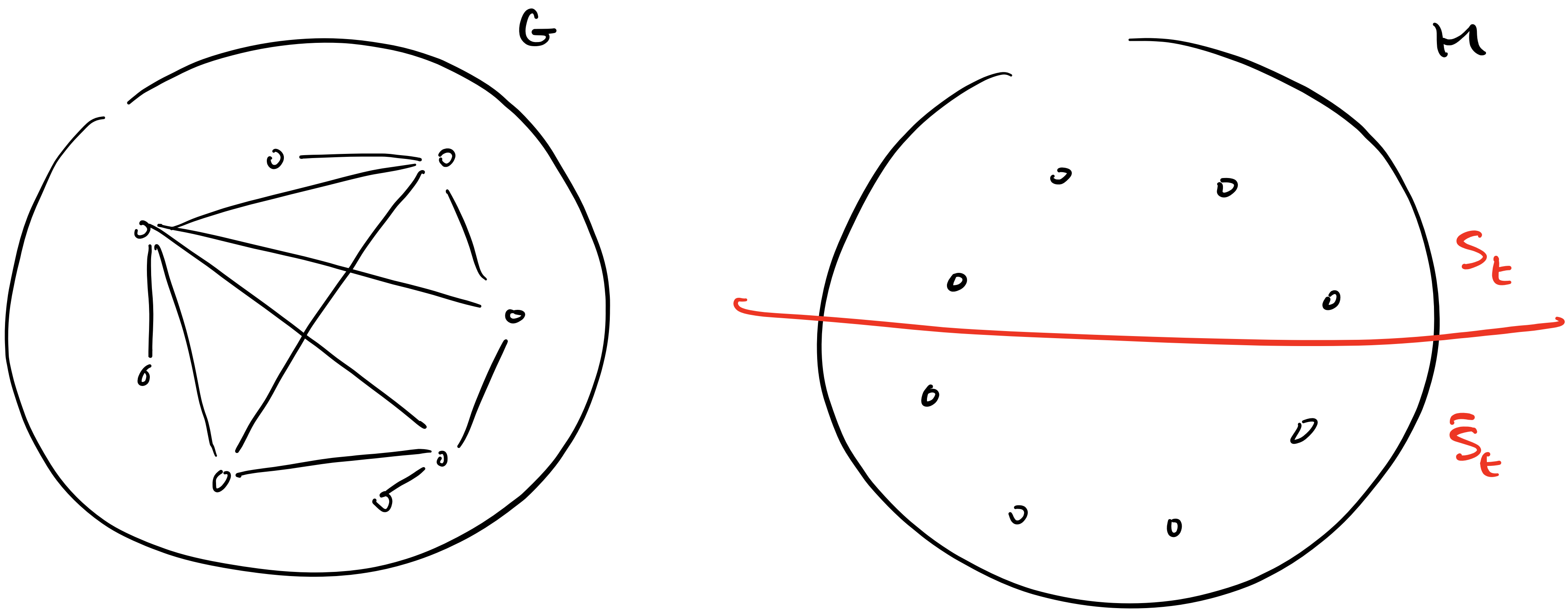
↪ <u>Theorem</u> [OSVV08]: ... $O(\log n)$-approx ... $O(\log^2 n)$-flows.

Prelude: Cut-Matching Games.

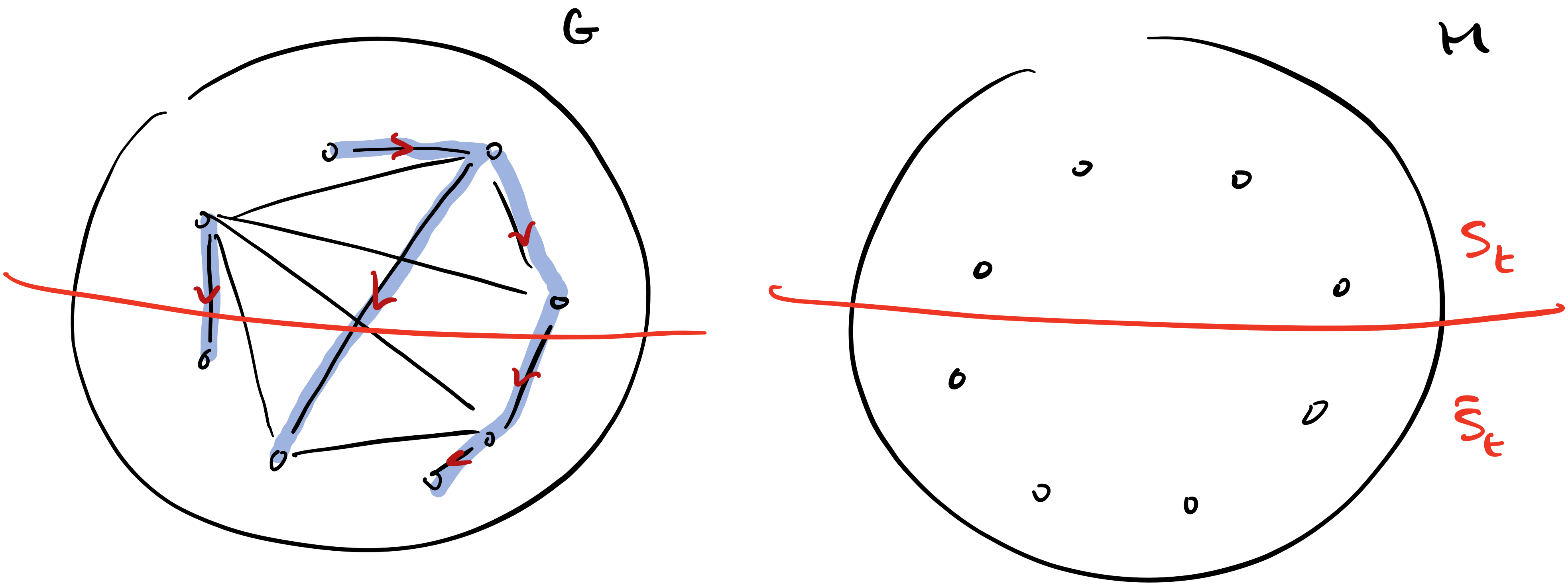


Prelude: Cut-Matching Games.



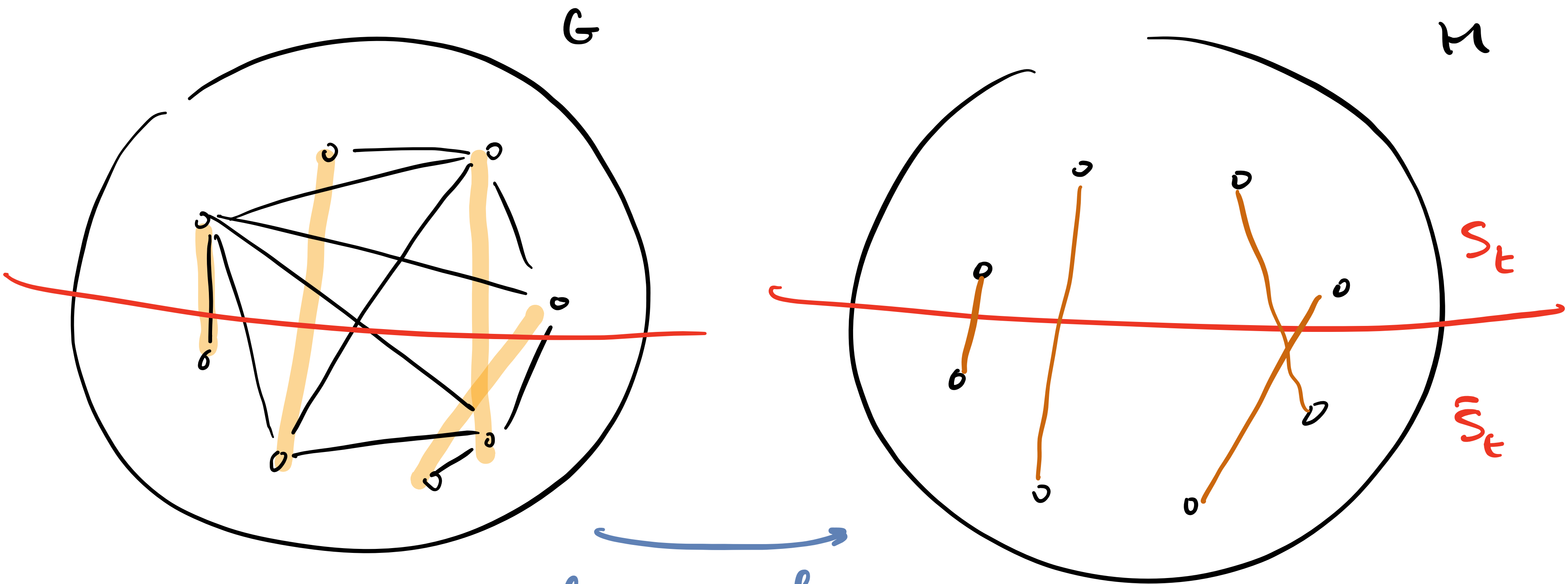
Cut Player: Finds a sparse bisection of H

Prelude: Cut-Matching Games.



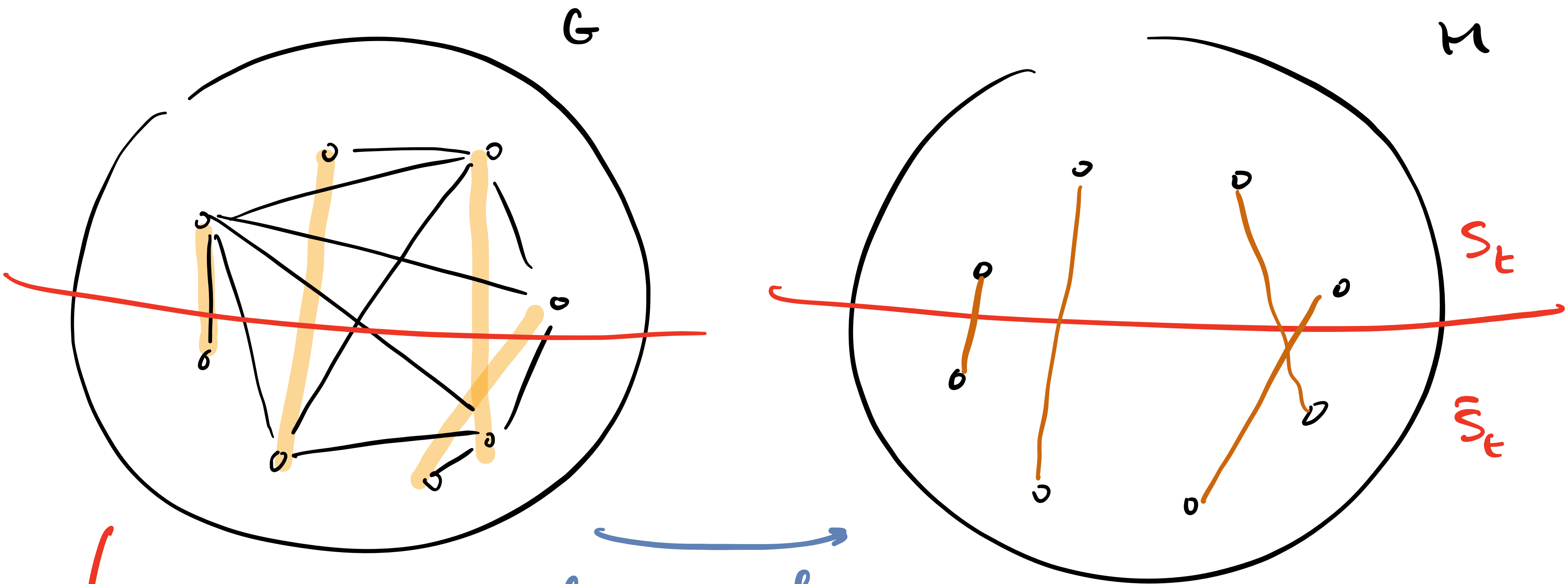
MATCHING PLAYER: tries to route flow in G across bisection from cut player.

Prelude : Cut-Matching Games.



If large flow routed ... add a matching M_t.

Prelude : Cut-Matching Games.



If large flow routed ... add a matching M_t.

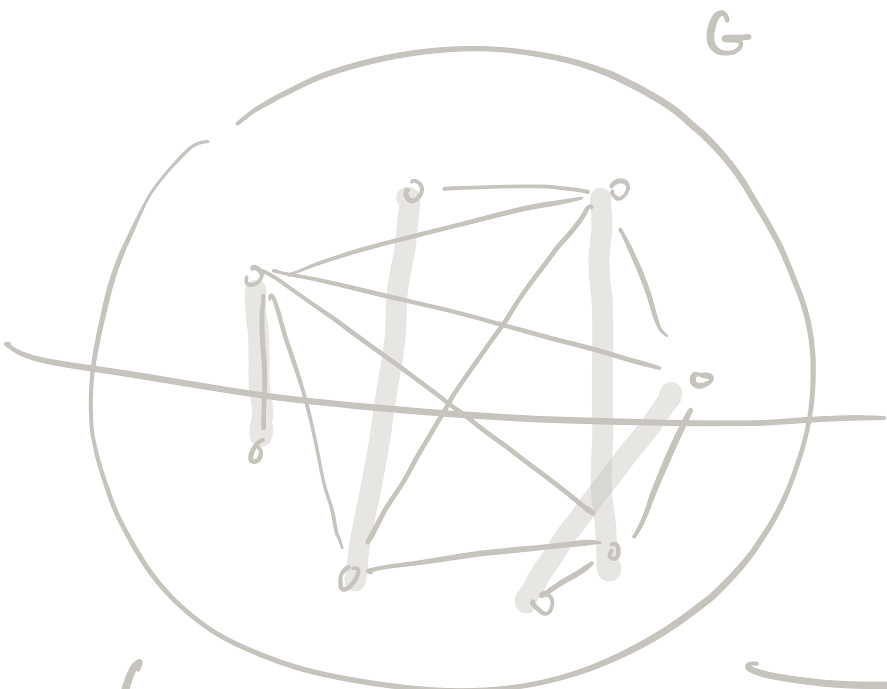
Otherwise...

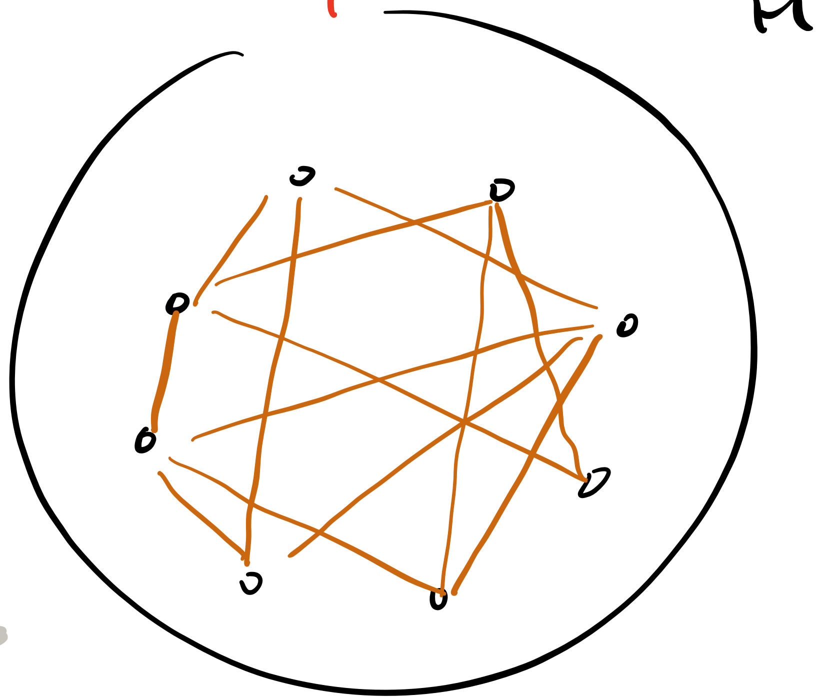
sparse cut!

Prelude : Cut-Matching Games.

(Spectral ...)

Expander !





If large flow

routed ... add a matching M_t.

Otherwise ...

Sparse cut! ⟶ After $O(\log^2 n)$ rounds ...

So okay... what are we going to do?

→ A new way to prove the result of OSVV08 bearing using tools from **convex optimization**.

So okay... what are we going to do?

→ A new way to prove the result of OSVV08 hearing using tools from convex optimization.

but why would you ever want to do this ?

A 3-Step Blueprint...

1. Derive a family of local convex surrogates.

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3. Compose local certificates to produce a global lower bound to expansion via boosting

read: mmwu..

A 3-Step Blueprint...

1. Derive a family of local convex surrogates.

2. Leverage convex duality to produce local certificates for expansion.

3. Compose local certificates to produce a global lower bound to expansion via boosting

→ let's go...

Local Convex Surrogates (Step #1)

Fact: Given any wtd graph $G=(V,E,w^G)$

$$\phi_G = \min_{x\in\mathbb{R}^V} \frac{\sum_{ij\in E} w^G_{ij}\cdot|x_i-x_j|}{\min_u \|x-u\mathbb{1}\|_1}$$

$\bar{\phi}_G(x)$

$\bar{q}_G$

Claim: $\forall\, x\in\mathbb{R}^V,\ s\in\mathbb{R}^V,\ s\perp\mathbb{1},\ \|s\|_\infty\le 1.$

$$\min_u \|x-u\mathbb{1}\|_1 \ge |\langle s,x\rangle|$$

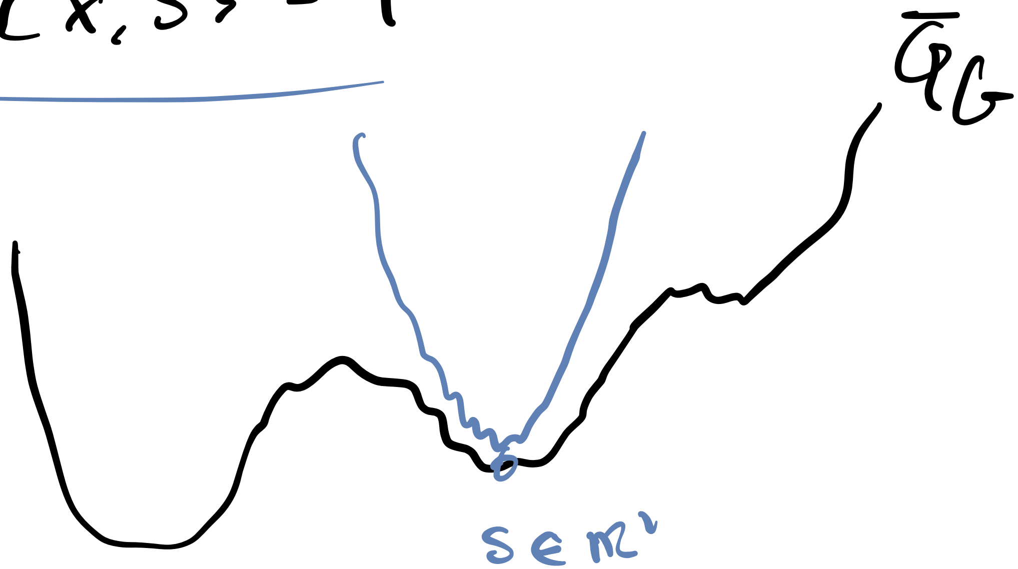
$$\longrightarrow \quad \bar{q}_G \le \min_{x\in\mathbb{R}^V} \frac{\sum_{ij\in E} w_{ij}\cdot|x_i-x_j|}{|\langle s,x\rangle|}$$

$$\left\{ \min_{x\in\mathbb{R}^V} \bar{q}_{G,s}(x) \;:\; \underbrace{s\in\mathbb{R}^V,\ \|s\|_\infty \le 1,\ s\perp \mathbb{1}}_{\text{"seeds"}} \right\}$$

$$\underbrace{\min_{x\in\mathbb{R}^V} \overbrace{\sum_{ij\in E} w^G_{ij}\cdot |x_i - x_j|}^{\bar{q}_{G,s}(x)} \quad \text{s.t.} \quad \langle x, s\rangle = 1}_{\bar{q}_{G,s}}$$

$A, B \subseteq U$ disjoint

$$s = \mathbb{1}_A - \frac{|A|}{|B|}\cdot \mathbb{1}_B$$



If you only takeaway one thing from this talk...

→ We can produce a family of local convex surrogates for expansion.

↪ Through this - leverage convex duality.

→ Furthermore the duals produce local certificates for expansion

↪ Boosting to produce a global lower bound.

→ ℓ_1-analogue of expansion still non-convex.!

→ The cut and matching player actions.

Local Dual Certificates (Step #2)

$$\min \bar{\varphi}_{G,s}(x) \xrightarrow{\text{dual}} \begin{aligned} &\max \ \alpha. \\ &\text{s.t.} \quad B^T f = s. \\ &\qquad \frac{1}{w_e} \cdot |f_e| \le \frac{1}{\alpha} \quad \forall e \in E. \end{aligned}$$

Claim: If D is the demand graph of f solved from the dual of $\varphi_{G,s}$ then

$$\bar{\varphi}_D \le \frac{1}{\alpha} \cdot \bar{\varphi}_G$$

Boosting Local Certificate (Step #3)

→ The flow-embedding statement is additive.

If you have $T > 0$ demand graphs $D_1 \dots D_T$ from solving $\overline{q}^*_{G,s_t}$ each embedding into G w/ congestion $\rho_1 \dots \rho_T > 0$ then

$$H = \frac{1}{T} \cdot \sum_{t=1}^{T} D_t$$

edge-wise!

"embeds" into G w/ congestion $\frac{1}{T} \cdot \sum_{t=1}^{T} \rho_t$.

Boosting Local Certificate (Step #3)

→ The flow-embedding statement is additive.

→ The bound is excellent when H is an expander.

- Given a sequence of seeds $s_1, \ldots, s_T \in \mathbb{R}^V$ outputting cut w/ smallest $\overline{\Phi}_{G, s_t}$ produces a $O\left(\frac{1}{\lambda_2(L_H)}\right)$-approximation to expansion.

Boosting Local Certificate (Step #3)

→ The flow-embedding statement is additive.

→ The bound is excellent when H is an expander.

→ Use boosting (MMWU)!

Produce a sequence of seeds $S_1 \dots S_T \in \mathbb{R}^V$ s.t. the demand graphs $D_1 \dots D_T$ average to produce

$$H = \frac{1}{T} \cdot \sum_{t=1}^{T} D_t$$ a $\Omega\left(\frac{1}{\log n}\right)$-expander.

↪ $T \leq O(\log^2 n)$.

Boosting Local Certificate (Step #3)

→ The flow-embedding statement is additive.

→ The bound is excellent when H is an expander.

→ Use boosting (MMWU)!

→ An $O(\log n)$-approximation for expansion using $O(\log^2 n)$ maximum flows!

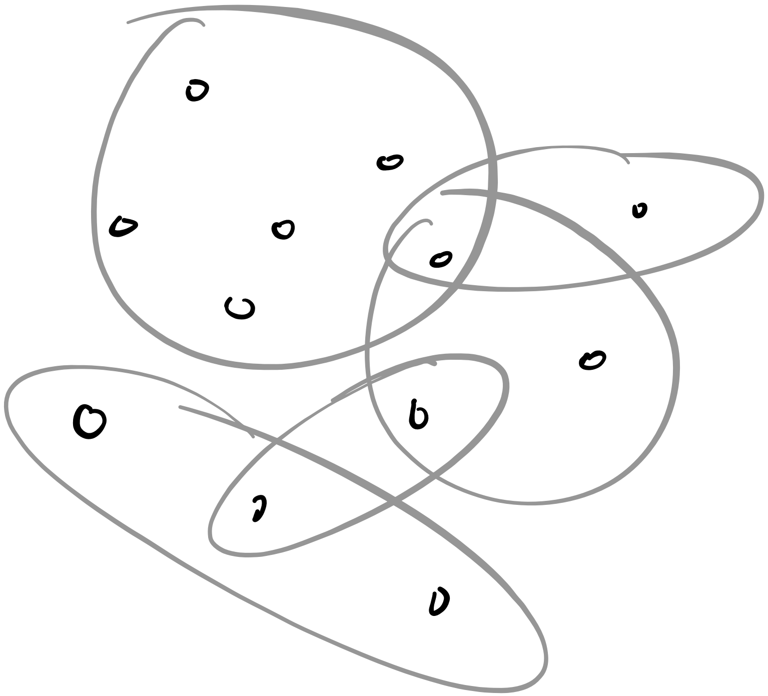
QUESTIONS?

Part 2 : The Present

Even though we give a new proof of CM-games, the algorithm is still a known result...

→ Applying this view to hypergraph partitioning

Hypergraph Partitioning

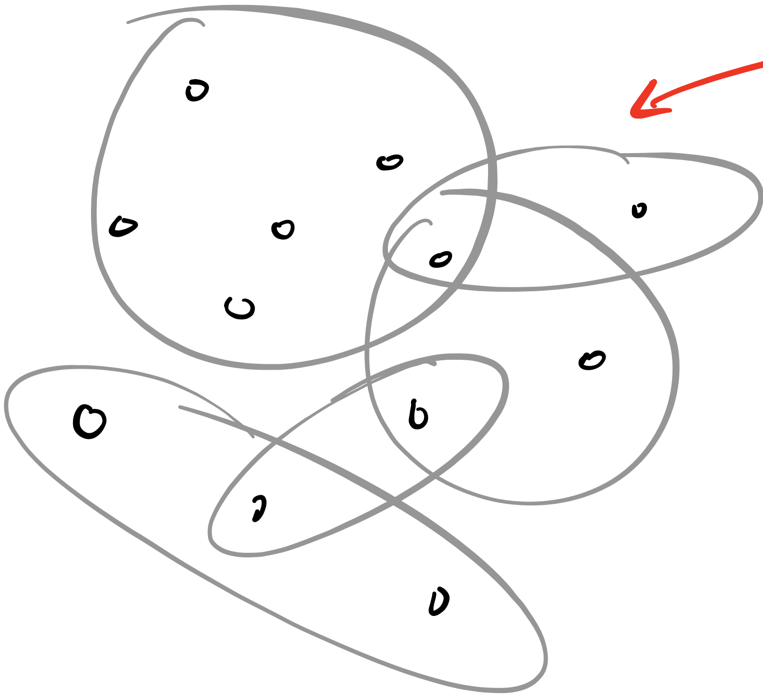


$G = (V, E, w^G, \mu)$ where

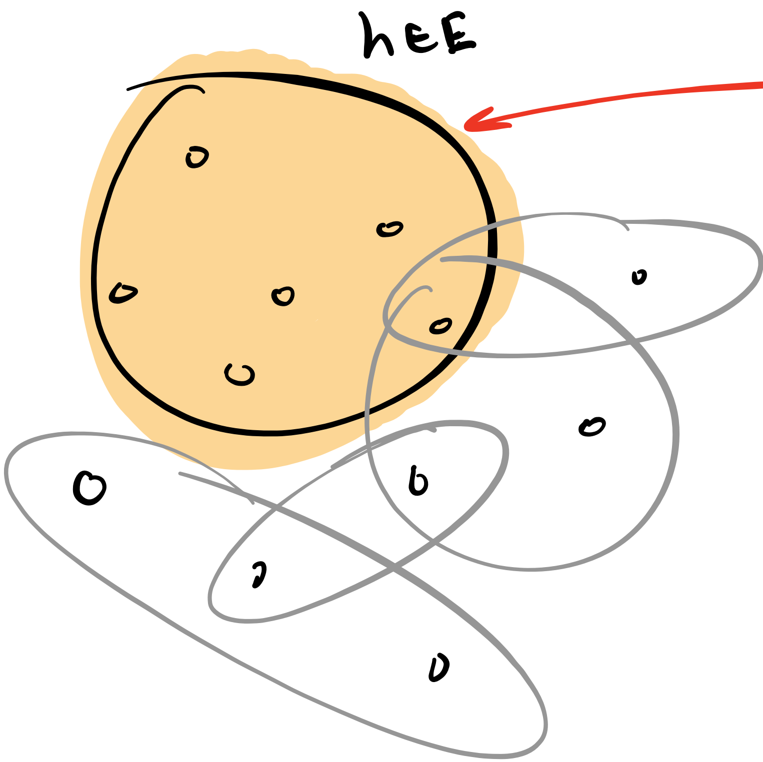
- Hyperedges are subsets of vertices $E \subseteq 2^V$.
- Hyperedges are weighted $w^G_h > 0 \quad \forall h \in E$
- Vertices have a positive measure $\mu_i > 0 \quad \forall i \in V$

Hypergraph Partitioning

Author network ?

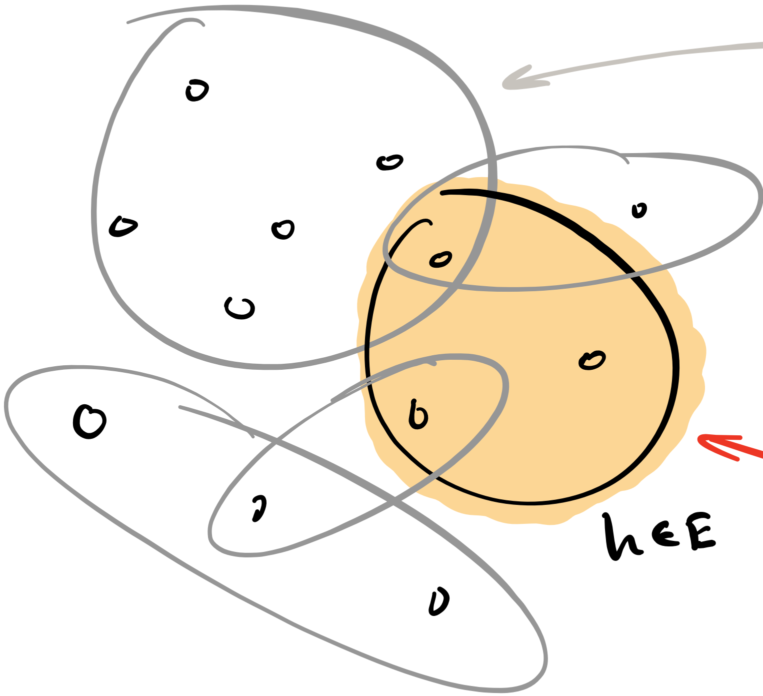


Hypergraph Partitioning



"How to start an NGO w/ zero experience". Garmer, Kottrage, Krishnan, Zheng, Chen

Hypergraph Partitioning



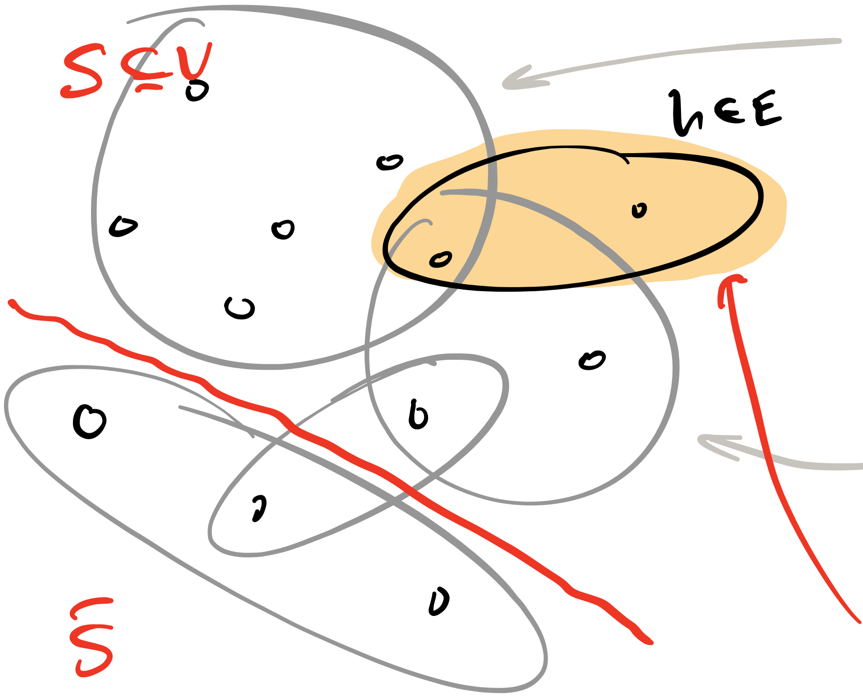
"How to start an NGO w/ zero experience"

Garner, Kottage, Krishnan, Zheng, Chen

"Maplestory is P-Space hard"

Lin, Zhou, Chen

Hypergraph Partitioning

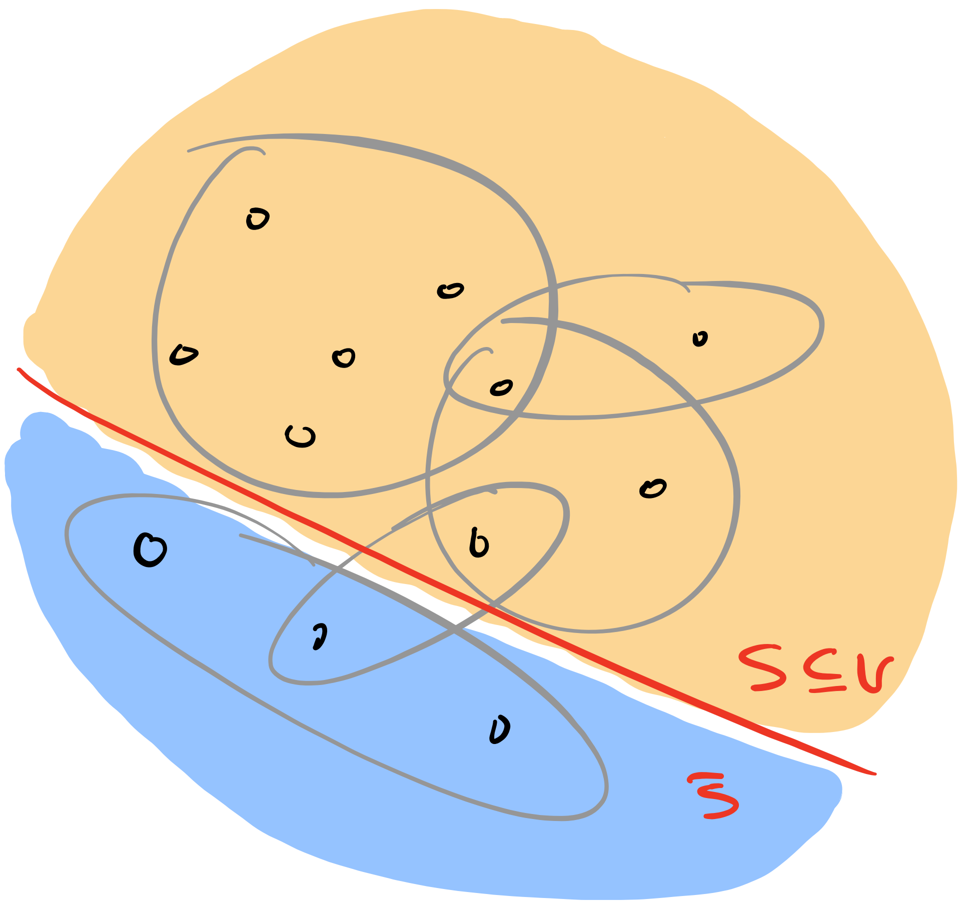


"How to start an NGO w/ zero experience" Garmer, Kottage, Krishnan, Zheng, Chen

"Maplestory is P-Space hard" Lin, Zhou, Chen

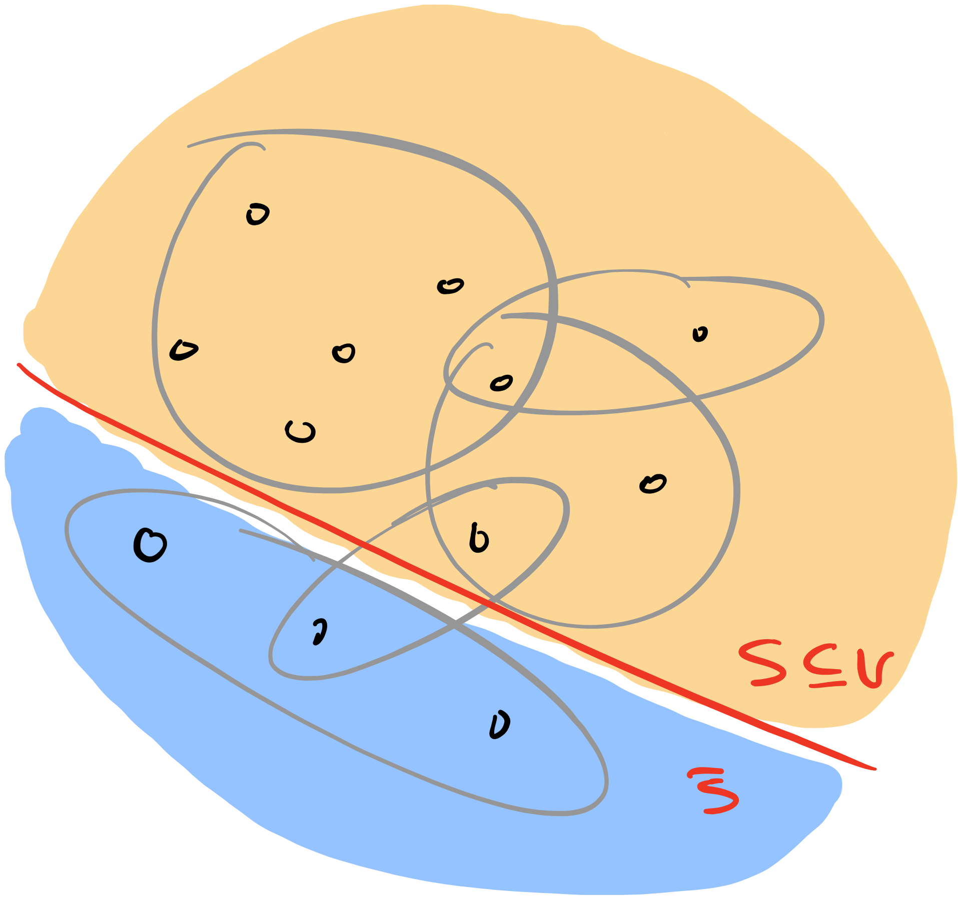
"Washing Machines via Reinforcement Learning" Choi, Chen

Hypergraph Partitioning



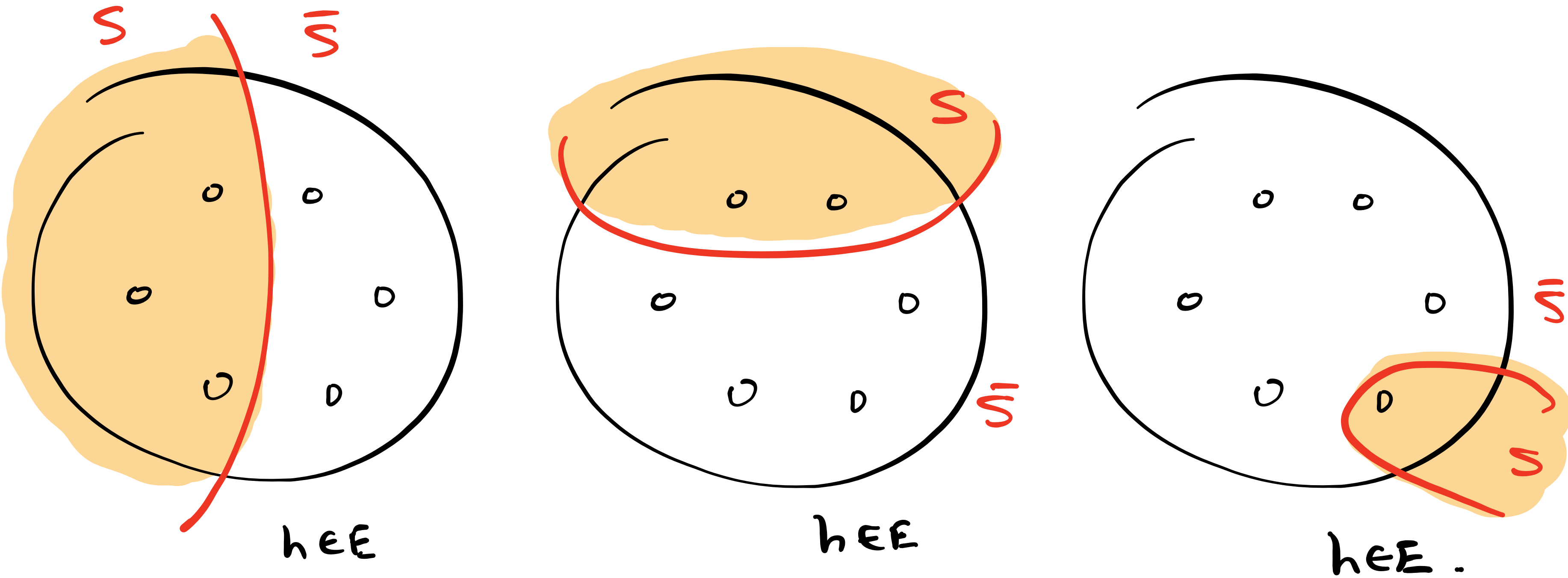
Antares's academic portfolio

Hypergraph Partitioning



Antares's academic portfolio

But Hyperedges can be cut in multiple ways.



→ Need to quantify the cost of cutting a hyperedge.

Polymatroidal Cut Functions

Def (polymatroid) : A hyperedge cut fn $\delta_h : 2^h \to \mathbb{R}_{\geq 0}$ is a polymatroid if there exists set functions $F_h^-, F_h^+ : 2^h \to \mathbb{R}_{\geq 0}$ such that

$$\delta_h(S) = \min \{ F_h^-(S), F_h^+(h \setminus S) \}.$$

AND ...

1. F_h^+, F_h^- are monotone submodular
2. $F_h^+(\emptyset) = F_h^-(\emptyset) = 0$

Polymatroidal Cut Functions

Def (polymatroid) : A hyperedge cut fn $\delta_h : 2^h \to \mathbb{R}_{\geq 0}$

is a pol... ...t functions

Submodular – $f : 2^h \to \mathbb{R}_{\geq 0}$, $\forall\, S, T \subseteq h$

$$f(S \cup T) \leq f(S) + f(T) - f(S \cap T)$$

monotone – $f : 2^h \to \mathbb{R}_{\geq 0}$, $\forall\, S \subseteq T \subseteq h$

$$f(S) \leq f(T)$$

F_h^-, F_h^+ ...

δ_v ... $)\}$.

AND ...

1. F_h^+, F_h^- are monotone submodular

Cheeger?

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Polymatroidal Cut functions

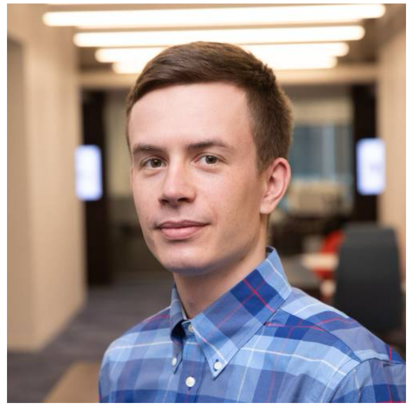
Def (polymatr... $\delta_v : 2^n \to \mathbb{R}_{\geq 0}$

is a pol... functions

F_h^-, F_h^+ ...

δ_v ...).

Great talk by Erasmo Tani.



AND ...

1. F_h^+, F_h^- are monotone submodular

Cheeger?

2. $F_h^+(\emptyset) = F_h^-(\emptyset) = 0$

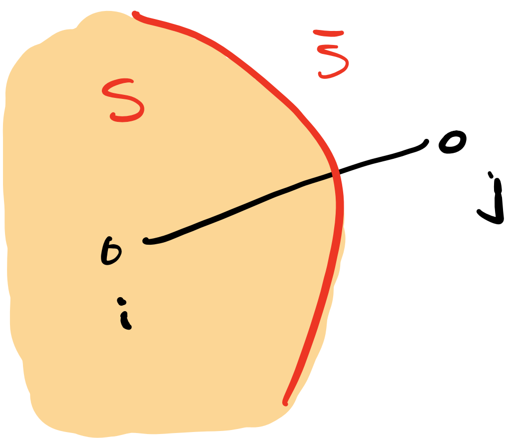
Why study polymatroidal cut fns

→ polymatroidal cut functions...

- **Expressive**: captures many typically considered (hyper)graph partitioning objectives.
- **Structured**: metric & flow techniques still apply to produce fast approx algs.

$$\delta_h(S) = \min \{ F_h^-(S), F_h^+(h \setminus S) \}$$

Undirected



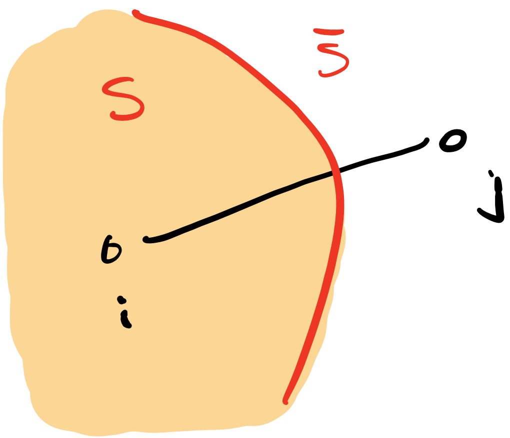
$\delta_{\{i,j\}}(S)$

$$= \min \{ \underbrace{|S \cap \{i,j\}|}_{F_h^-(S)}, \underbrace{|\bar{S} \cap \{i,j\}|}_{F_h^+(h \setminus S)} \}$$

$\longrightarrow F_h^-(S) = F_h^+(S)$

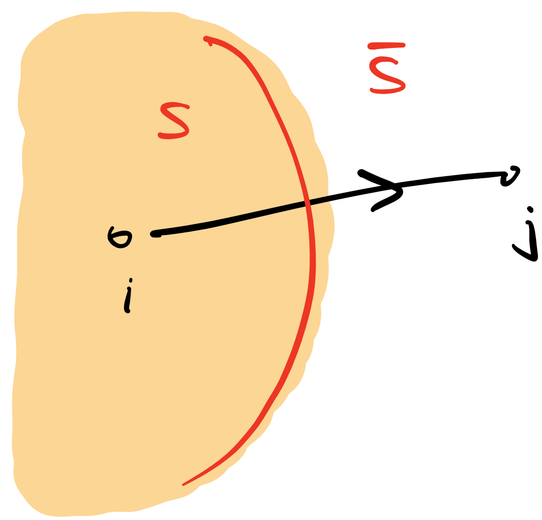
$$\delta_h(S) = \min\{F_h^-(S),\ F_h^+(h \setminus S)\}$$

Undirected



$$\delta_{\{i,j\}}(S) = \min\{|S \cap \{i,j\}|,\ |\bar{S} \cap \{i,j\}|\}$$

Directed



$$\delta_{(i,j)}(S) = \min\{\underbrace{|S \cap \{i\}|}_{F_h^-(S)},\ \underbrace{|\bar{S} \cap \{j\}|}_{F_h^+(h \setminus S)}\}$$

$F_h^-(S) \neq F_h^+(S)$

Minimum Ratio Cut.

INPUT: Hypergraph $G = (V, E, w^G, \mu)$ w/ polymatroidal cut fns. $\{\delta_h\}_{h \in E}$

OUTPUT: $S \subseteq V$ minimizing ratio cut objective

$$\Psi_G(S) := \frac{\sum_{h \in E} w_h^G \cdot \delta_h(S)}{\min\{\mu(S), \mu(\bar{S})\}}$$

Denote $\Psi_G = \min_{S \subseteq V} \Psi_G(S) \ldots$

Our result...

<u>Theorem</u> [C, Orecchia, Tani 22]: $\exists$ a randomized algorithm $\mathcal{A}$ which outputs an $O(\log n)$-approximation to minimum ratio cut. i.e. $S \subseteq V$ s.t.

$$\Phi_G \leq \Phi_G(S) \leq O(\log n) \cdot \Phi_G.$$

Furthermore ...

Our result...

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2. If $\{\delta_h\}_{h \in E}$ not $\longrightarrow$ $O(\log^3 n)$ ...

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A Brief Perspective on generalizing...

→ <u>Step #1</u> : family of convex surrogates

$$\Phi_G = \min_{S \subseteq V} \frac{\sum_{h \in E} w_h^G \cdot \delta_h(S)}{\min\{\mu(S), \mu(\bar{S})\}}$$

A Brief Perspective on generalizing ...

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$$= \min_{x \in \mathbb{R}^V} \frac{\sum_{h \in E} w_h^G \cdot \bar{\delta}_h(x)}{\min_u \|M(x - u\mathbb{1})\|_1}$$

Lovász extension

diag(μ)

A Brief Perspective on generalizing...

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$$= \min_{x \in \mathbb{R}^V} \frac{\sum_{h \in E} w_h^G \cdot \bar{\delta}_h(x)}{\min_u \| m(x - u\mathbb{1})\|_1}$$

$$\Phi_{G,s} = \begin{array}{ll} \min & \sum_{h \in E} w_h^G \cdot \bar{\delta}_h(x) \\ \text{s.t.} & \langle s, x \rangle = 1 \\ & x \in \mathbb{R}^V \end{array}$$

cut improvement.

A Brief Perspective on generalizing ...

→ **Step #1** : family of convex surrogates

$$\Phi_G = \min_{S \subseteq V} \frac{\sum_{h \in E} w_h^G \cdot \delta_h(S)}{\min\{\mu(S), \mu(\bar{S})\}}$$

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Flow improvement

$$\Phi_{G,s} = \begin{array}{ll} \min & \sum_{h \in E} w_h^G \cdot \bar{\delta}_h(x) \\ \text{s.t.} & \langle s, x \rangle_\mu = 1 \\ & x \in \mathbb{R}^V \end{array} \xrightarrow{\text{dual}} \begin{array}{ll} \max & \alpha \\ \text{s.t.} & \sum_{h \in E} f_h = s \\ & \frac{1}{w_h} \cdot f_h \in \frac{1}{\alpha} \cdot B(\delta_h) \quad \forall h \in E \\ & f_h \in \mathbb{R}^h \quad \forall h \in E \end{array}$$

Hypergraph flow

Base polytope

Questions?

Thank You!

SECRET CONTENT

PART 3 : THE FUTURE

The regret bound from MMWU.

$$\lambda_2(L_H) \geq \frac{1-\epsilon\eta}{T} \cdot \sum_{t=1}^{T} \langle L_{D_t}, X_t \rangle - \frac{\log n}{\eta T}$$

Defn: $S, T \subseteq V$ disjoint are Δ-separated if.

$$\|v_i - v_j\|^2 \geq \Delta \cdot \frac{1}{\mu(S) \cdot \mu(T)} \cdot \sum_{ij \in V} \|v_i - v_j\|^2$$

$$\forall i \in S, j \in T.$$

Nonuniform demands?

1. Fast algorithms for non-uniform sparsest cut

2. $\Omega\left(\frac{1}{\sqrt{\log n}}\right)$ - separated sets for non-uniform sparsest cut w/ product demands.

   ↪ In $O(\log^k n)$ flows.

3. Polytime - worst case approximations for non-uniform sparsest cut.

Notion of Flow Embedding

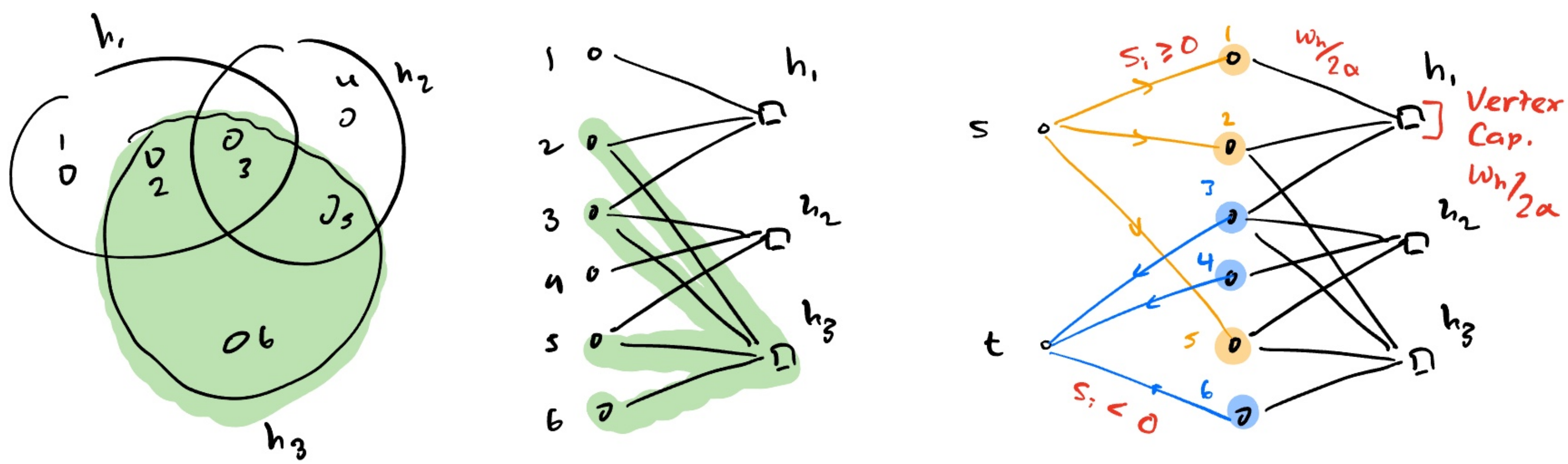
If $H = (V, E_H, w^H)$ is a directed graph and $G = (V, E_G, w^G, \mu)$ is a hypergraph equipped w/ polymatroidal cut functions $\{\delta_h\}_{h \in E}$ then $H \preceq_\rho G$ if $\exists$ a hypergraph flow $\{f_h^{(u,v)}\}_{h \in E_G, (u,v) \in E_H}$ s.t.

1. The flow is routable in G:

$$\sum_{h \in E_G : h \ni u} f_h^{(u,v)}(u) = w_{uv}^H \quad \text{and} \quad \sum_{h \in E : h \ni v} f_h^{(u,v)}(v) = -w_{uv}^H$$

2. The flow has congestion $\leq \rho$

$$\sum_{(u,v) \in E_H} \left(f_h^{(u,v)}\right)_+ \in \rho \cdot w_h^G \cdot P_{sym}(F_h^-) \quad \text{and} \quad \sum_{(u,v) \in E_H} \left(f_h^{(u,v)}\right)_- \in \rho \cdot w_h^G \cdot P_{sym}(F_h^+)$$



Hypergraph

$G = (V, E, w)$

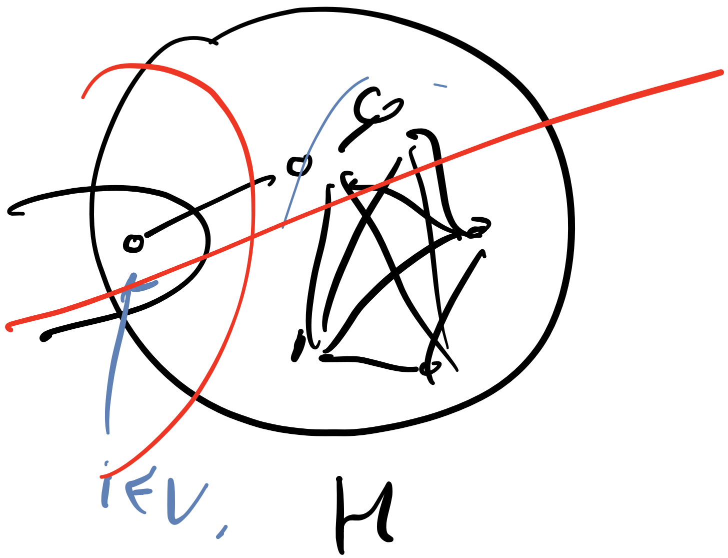
Factor graph $\hat{G}$

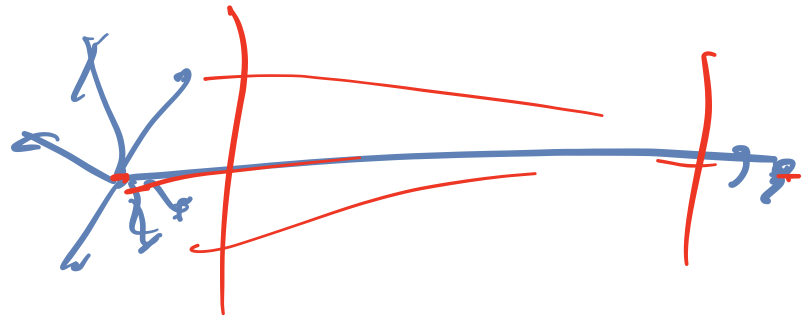
$E(\hat{G}) = \{ (i,h) : i \in h \ \forall h \in E \}$

Flow network given seed

$$s = \begin{pmatrix} +1 & +1 & -1 & -1 & +1 & -1 \end{pmatrix}^T$$

$i=1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$





$$\delta_n : 2^n \to \mathbb{R}^V \implies P_{sym} = \{ x \in \mathbb{R}^V : \sum_{i \in S} |x_i| \le \delta_n |S| \}.$$

$$\|x\|_{\delta_n} := \max_{\substack{y \in P_{sym} \\ y \perp \mathbb{1}}} \langle y, x \rangle.$$

$$\min_{u} \max_{y \in P_{sym}} \langle y, x - u\mathbb{1} \rangle.$$

$$\|x\|_{\delta_n} \le \underbrace{\delta_n(x)} \le \underbrace{? \cdot \|x\|_{\delta_n}}$$